Lagrangian Approximations and Collocation Method for Solution of Integral Equations of the First Kind

Olexandr Polishchuk
Laboratory of Modeling and Optimization of Complex Systems
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics National Academy of Sciences of Ukraine
Lviv, Ukraine
od_polishchuk@ukr.net

***Abstract*—This article is dedicated to research of approximation properties of Lagrangian finite elements in Hilbert spaces of functions defined on surfaces in three-dimensional space. The conditions are determined for convergence of collocation methods for solving Fredholm integral equation of the first kind for simple layer potential that is equivalent to Dirichlet problem for Laplace equation in $\mathbf{R}^3$. Estimation is determined for the error of approximate solution of this problem obtained using potential theory methods.**

***Keywords*—potential; integral equation; well-posed solvability; Lagrange interpolation; collocation method; convergence**

I. INTRODUCTION

Many physical processes (e.g. diffusion, heat flux, electrostatic field, perfect fluid flow, elastic motion of solid bodies, groundwater flow, etc.) are modeled using boundary value problems for Laplace equation [1]. The powerful tools for solving such problems are potential theory methods, especially in the case of tired boundary surface or complex shape surface [2]. In number of cases, application of potential theory methods requires solving Fredholm integral equation of the first kind. In particular, one of the cases is solving Dirichlet problem in the space of functions with normal derivative jump on crossing boundary surface using simple layer potential [3]. The need to solve integral equations of the first kind also arises when the sum of simple and double layer potentials is used to solve the double-sided Dirichlet or Neumann problem [4] or double-sided Dirichlet-Neumann problem [5] in the space of functions that, same as their normal derivatives, have jump on crossing boundary surface. Many systems of integral equations for the simple and double layer potentials that are equivalent to mixed boundary value problems for Laplace equation, also contain integral equations of the first kind [6].In general, researches of projection methods convergence mainly focus on solving integral equations of the second kind. Defining well-posed solvability conditions for integral equations of the first kind that are equivalent to boundary value problems for Laplace's equation in Hilbert spaces [7, 8] allows us to use projection methods for numerical solution of such equations, thus avoiding resource-consuming regularization procedures. In [9, 10] convergence conditions are defined for the series of projection methods for solving Fredholm integral equation of the first kind for simple layer potential that is equivalent to three-dimensional Dirichlet problem for Laplace equation while approximating desired potential density with complete systems of orthonormal functions. However, if boundary surface has a complex shape usage of such approximations poses considerable difficulties for practical implementation of numerical methods. In this case, finite elements of different types should be used for approximation of desired potential densities. Derived approximations, among other things, allow us to create effective algorithms for singularities removal in kernels and desired integral equation densities [11].

The purpose of the paper is to define convergence conditions of collocation method for approximate solution of Fredholm integral equations of the first kind by the example of integral equation for the simple layer potential that is equivalent to Dirichlet problem for Laplace equation using approximation of desired potential density with systems of Lagrangian finite elements of different orders.

II. LAGRANGIAN APPROXIMATIONS

Let us $S=[0,a]\times[0,b]\subset R^2$. Construct in the domain S a rectangular grid S_h with the steps $h_1=a/n$ and $h_2=b/k$, $n,k=1,2,...$ Assign to each element of the grid S_h of domain S

$$P_{ij}=[h_1 i,h_1(i+1)]\times[h_2 j,h_2(j+1)],$$

$$i=\overline{0(1)(n-1)}\,,\; j=\overline{0(1)(k-1)},$$

a smaller rectangular grid P_{ij}^{ε} with thesteps $\varepsilon_1=h_1/m$ and $\varepsilon_2=h_2/m$. Denote $S_{h,\varepsilon}=\bigcup\limits_{i,j}P_{ij}^{\varepsilon}$ andassociate with the set of nodes $S_{h,\varepsilon}$ a system of piecewise polynomial functions

$$\{L_{pt}(\xi)\}_{p=0}^{mn}{}_{t=0}^{mk}, \tag{1}$$

satisfying conditions

$$L_{pt}(\xi_{ls})=\delta_{pl}\delta_{ts}, \operatorname{supp}\{L_{pt}(\xi)\}=\widetilde{P}_{pt},$$

$$\widetilde{P}_{pt}=\{\bigcup_{i,j}P_{ij}:\xi_{pt}\in P_{ij}\}, \xi_{ls}\in\widetilde{P}_{pt}, \tag{2}$$

where δ_{pl} is the Kronecker symbol.

Functions (1)-(2) form a system of Lagrangian finite elements of m-th degree in $H^m(S)$.Denote by $U_L^{N_1}$ the linear shellof this system, $N_1=(1+mn)(1+mk)$. It is obvious that the restriction of system (1)-(2) onto an arbitrary rectangle P_{ij} of the grid S_h is a basis in the space of polynomials $P^m(P_{ij})$ of degree not higher than m, defined on P_{ij}. Then

$$U_B^N \subset U_L^{N_1}, \qquad (3)$$

where U_B^N is the linear shellof the system of B-splines of m-th degree defined on S_h [12].

Choose the extension operator $p_L^{N_1}: V_L^{N_1} \to U_L^{N_1} \subset H^m(S)$, where $V_L^{N_1} \subset R^{N_1}$, in the form

$$p_L^{N_1}\mathbf{v}_L^{N_1} = \sum_{i=0}^{mn}\sum_{j=0}^{mk} v_{N_1}^{(i,j)} L_{ij}(\xi). \qquad (4)$$

Then, byvirtueoftheembedding (3), thereexistsarestrictionoperator $r_L^{N_1}: H^m(S) \to V_L^{N_1}$ suchthatapproximations $(V_L^{N_1}, p_L^{N_1}, r_L^{N_1})$ of the space $H^m(S)$ are convergent and valid the estimates

$$\left\| v - p_L^{N_1} r_L^{N_1} v \right\|_{H^t(S)} \le \widetilde{C} h^{\sigma - t} \|v\|_{H^\sigma(S)}, \qquad (5)$$

where $0 \le t \le \sigma \le m+1$, $t \le m$, and constant $\widetilde{C} > 0$ does not depend on v. Thus, it is proved

Lemma1. Thereisarestrictionoperator $r_L^{N_1}: H^m(S) \to V_L^{N_1}$ suchthatapproximations $(V_L^{N_1}, p_L^{N_1}, r_L^{N_1})$ of the space $H^m(S)$ are convergent and valid the estimates (5).

Assume that surface $\Gamma = \bigcup_{l=1}^{M} \Gamma_l$ is m-smooth surface in R^3 [13]. Constructineach domain S_l the rectangulargrid S_l^h with the steps $h_1^{(l)} = a_l / n_l$ and $h_2^{(l)} = b_l / k_l$ and set on each element P_{ij}^l of the grid S_l^h a smaller grid with the steps $\varepsilon_1^{(l)} = h_1^{(l)} / m$ and $\varepsilon_2^{(l)} = h_2^{(l)} / m$, $l = \overline{1,M}$. Define analogously to (1), (2) in each grid domain $S_l^{h,\varepsilon} = \bigcup_{i,j} P_{ij}^{l,\varepsilon}$ the system of Lagrangian finite elements

$$\{L_{ij}^{(l)}(\xi^{(l)})\}_{i=0}^{n_l m}\,{}_{j=0}^{k_l m}, \xi^{(l)} \in S_l, l = \overline{1,M}.$$

Assign to the family $S_l^{h,\varepsilon}$ the grid $\Gamma_{h,\varepsilon} = \bigcup_{l=1}^{M} \tau_l^{-1}(S_l^{h,\varepsilon})$ on the surface Γ, where $\tau_l^{-1}(P_{ij}^{l,\varepsilon})$ are the elements of the grid $\Gamma_{h,\varepsilon}$, $l = \overline{1,M}$. Denote by T_l the set of nodes of the grid $S_l^{h,\varepsilon}$, $l = \overline{1,M}$, $T = \bigcup_{l=1}^{M} T_l$. We number all elements of the set T with the cross-cutting index $t = \overline{1,K}$, $K = \sum_{l=1}^{M} K_l$, $K_l = (1 + n_l m)(1 + k_l m)$, and put in correspondence to each node x_p of the grid $\Gamma_{h,\varepsilon}$ the set of elements

$$P_p^* = \{P_{ij}^{l,\varepsilon} \subset \bigcup_{l=1}^{M} S_l^{h,\varepsilon} : x_p \in \tau_l^{-1}(P_{ij}^{l,\varepsilon})\},$$

element

$$\widetilde{P}_p = \{\bigcup_{i,j} \tau_l^{-1}(P_{ij}^{l,\varepsilon}), P_{ij}^{l,\varepsilon} \in P_p^*, l = \overline{1,M}\},$$

the set of indexes

$$T_p^* = \{t \in T : \tau_l^{-1}(P_{ij}^{l,\varepsilon}) = x_p, \xi_t^{(l)} \in P_l^{h,\varepsilon}\ l = \overline{1,M}\},$$

and function

$$\widetilde{L}_p(x) = \sum_{t \in T_p^*} L_t^l(\tau_l(x)), x \in \Gamma_l, \operatorname{supp}\{\widetilde{L}_p(x)\} = \widetilde{P}_p, \qquad (6)$$

$$L_t^{(l)}(\xi^{(l)}) \in \{L_{ij}^{(l)}(\xi^{(l)})\}_{i=0}^{n_l m}\,{}_{j=0}^{k_l m}.$$

Denote by $\widetilde{r}_L^{N_L}$ the restriction operator from $H^m(\Gamma)$ into the finite dimensional space $V_L^{N_L}$ and by $\widetilde{r}_L^{N_l}$ – its restriction to $H^m(\Gamma_l)$, i.e.

$$\widetilde{r}_L^{N_L} = \{\widetilde{r}_L^{N_l}\}_{l=1}^{M}, \widetilde{r}_L^{N_l} u_l(x) = r_L^{K_l} v_l(\xi), \qquad (7)$$

where $r_L^{K_l}$ is the restriction operator from $H^m(S_l)$ into the corresponding finite dimensional space $V_L^{K_l}$, $l = \overline{1,M}$, and N_L is the number of nodes in the grid $\Gamma_{h,\varepsilon}$.

The extraction operator $\widetilde{p}_L^{N_L}$ from $V_L^{N_L}$ into the linear shell $U_L^{N_L}$ of the system $\{\widetilde{L}_p(x)\}_{p=1}^{N_L}, U_L^{N_L} \subset H^m(\Gamma)$, introduce by formula

$$(\widetilde{p}_L^{N_L} \mathbf{u}_L^{N_L})(x) = \sum_{i=1}^{N_L} u_N^{(i)} \widetilde{L}_i(x), \mathbf{u}_L^{N_L} \in V_L^{N_L}. \qquad (8)$$

From Lemma 1 follows that i.e. approximations $(V_L^{N_L}, \widetilde{p}_L^{N_L}, \widetilde{r}_L^{N_L})$ of the space $H^m(\Gamma)$ are convergent. Further from estimate (5) we obtain

$$\left\| u - \widetilde{p}_L^{N_L} \widetilde{r}_L^{N_L} u \right\|_{H^t(\Gamma)}^2 \le \widetilde{C}^2 h^{2(\sigma - t)} \|u\|_{H^\sigma(\Gamma)}^2, \qquad (9)$$

where $0 \le t \le \sigma \le m+1$, $t \le m$, and $p_L^{N_l}$ is a similar to (4) extension operator from $V_L^{N_l}$ into $H^m(S_l)$, constant $\widetilde{C} > 0$ does not depend on u and $h = \max_{1 \le l \le M}\{h_1^{(l)} h_2^{(l)}\}$. Thus, it is proved

Lemma 2. Thereisarestrictionoperator $\widetilde{r}_L^{N_L}: H^m(\Gamma) \to V_L^{N_L}$ suchthatapproximations $(V_L^{N_L}, \widetilde{p}_L^{N_L}, \widetilde{r}_L^{N_L})$ of the space $H^m(\Gamma)$ are convergent and valid the estimates (9).

III. COLLOCATION METHOD

Let us denote $G' = R^3 \setminus \overline{G}$ and introduce in G and G' the Sobolev spaces [13]

$$H^m(G) = \{v \in L_2(G) : \partial^\alpha v \in L_2(G), |\alpha| \le m\},$$

$$W^m(G') = \{v \in D'(G') : (1+r^2)^{(|\alpha|-1)^2} \partial^\alpha v \in L_2(G'), |\alpha| \le m\},$$

where $m \ge 0$, and $r = (\sum_{i=1}^{3} x_i^2)^{1/2}$, $x = (x_1, x_2, x_3) \in R^3$.

Consider the next boundary value problem: to find function

$$v \in H^{m+1}_{\Gamma,\Delta=0} = \{v \in H^{m+1}(G) \cup W^{m+1}(G') : \quad (10)$$
$$v|_{\Gamma_{\text{int}}} = v|_{\Gamma_{ext}}, \Delta v(x) = 0, x \in G, G'\}$$

satisfying condition

$$v|_\Gamma = f,\ f \in H^{m+1/2}(\Gamma). \quad (11)$$

In [14] was proved

Theorem 1. Problem (10)-(11) has one and only one solution.

We will search a solution of the problem (10) - (11) in the form of simple layer potential

$$v(x) = \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} d\Gamma_y,\ x \in G, G'.$$

The unknown potential density is determined from the equation

$$(\mathrm{Au})(x) \equiv \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} d\Gamma_y = f(x),\ x \in \Gamma. \quad (12)$$

Operator A is an isomorphism of $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ [14]. Then from the Banach theorem follows the validity of inequalities

$$\alpha_s \|u\|_{H^s(\Gamma)} \le \|Au\|_{H^{s+1}(\Gamma)} \le \beta_s \|u\|_{H^s(\Gamma)}, \quad (13)$$

in which constants α_s and β_s, $0 < \alpha_s \le \beta_s$, does not depend on $u \in H^s(\Gamma)$.

To simplify the presentation, we assume that for approximation of unknown potential density $u \in H^m(\Gamma)$, $m \ge 0$, of equation (10) a system of linearly independent functions $\{\varphi_i\}_{i=1}^{\infty}$ is chosen, U_N is a linear shell of the system $\{\varphi_i\}_{i=1}^{N}$, $r_N : H^m(\Gamma) \to V_N$, $p_N : V_N \to U_N$ are the restriction and extraction operators. Denote by X_N the set of pairwise different points belonging to the surface Γ

$$X_N = \{x_j\}_{j=1}^{N},\ x_j \in \Gamma,\ j = \overline{1,N},$$

and introduce in $H^{m+1}(\Gamma)$ restriction operator $s_N : H^{m+1}(\Gamma) \to \Phi_N$ by formula

$$(s_N f)_j = f(\tilde{y}_j) \quad (14)$$

in which

$$\tilde{y}_j \in \{\tilde{y} \in \delta(y_j) : |f(\tilde{y})| = \min_{y \in \delta(y_j)} |f(y)|,\ y_j \in X_N\}, \quad (15)$$

$$\delta(y_j) = \{y \in \Gamma : |y - y_j| < \delta\},$$

in particular $\rho(y^*, \delta(y_j)) > 0$ for arbitrary $y^* \in X_N$, $y^* \ne y_j$, $j = \overline{1,N}$.

If $f \in C(\Gamma)$, then operator s_N can be defined as usual

$$(s_N f)_j = f(y_j),\ y_j \in X_N, \quad (16)$$

i.e. $\tilde{y}_j = y_j$, $j = \overline{1,N}$. It is easy to see that, with this choice of operator s_N, a system of linear algebraic equations

$$\mathbf{A}_N^c \mathbf{u}_N = s_N f,\ \mathbf{A}_N^c = s_N A p_N,\ \mathbf{u}_N \in V_N, \quad (17)$$

implements the collocation method of solving the equation (12). The set X_N is called a set of collocation points.

Denote $Y_N = \{\tilde{y}_j\}_{j=1}^{N}$ andconsider the system of functions

$$r_j(x) = \frac{1}{|x - \tilde{y}_j|},\ \tilde{y}_j \in Y_N,\ j = \overline{1,N}.$$

From the choice of the set X_N and conditions (15) follows that the functions of system $\{r_j(x)\}_{j=1}^{N}$ are linearly independent [15].

Define in $L^\infty(\Gamma)$ the family of linear continuous functionals

$$l_j(\varphi) = \int_\Gamma \varphi(x) r_j(x) d\Gamma_x,\ \varphi \in L^\infty(\Gamma),\ j = \overline{1,N}.$$

Denote by $Ker(l_j)$ the zero subspace of functional l_j in $L^\infty(\Gamma)$, i.e. $Ker(l_j) = \{\varphi \in L^\infty(\Gamma) : l_j(\varphi) = 0\}$ andsupposethat $K_N = \bigcap_{j=1}^{N} Ker(l_j)$. The degeneracy of matrix $\mathbf{A}_N^c$ is equivalent to the linear dependence of its rows or columns, that is, the existence of such sets $\boldsymbol{\alpha}_N = \{\alpha_i\}_{i=1}^{N} \in R^N$ or $\boldsymbol{\beta}_N = \{\beta_j\}_{j=1}^{N} \in R^N$, $\sum_{i=1}^{N} \alpha_i^2 > 0$, $\sum_{j=1}^{N} \beta_j^2 > 0$, that

$$\int_\Gamma (\sum_{i=1}^{N} \alpha_i \varphi_i(x)) r_j(x) d\Gamma_x = 0,\ j = \overline{1,N}, \quad (18)$$

or

$$\int_\Gamma \varphi_i(x) (\sum_{j=1}^{N} \beta_i r_j(x)) d\Gamma_x = 0,\ i = \overline{1,N}. \quad (19)$$

Implementation of equations (18), (19) is only possible if $K_N \cap U_N \ne 0$. From this follows sufficient conditions for the invertibility of matrix $\mathbf{A}_N^c$, which we formulate in the next statement.

Lemma3. Let us the system of linearly independent functions $\{\varphi_i\}_{i=1}^N$ is chosen for the approximate solution of equation (12) and determined the set of collocation points X_N (and, consequently, the set K_N is defined). Then, if

$$K_N \cap U_N = 0\,, \qquad (20)$$

then the matrix $\mathbf{A}_N^c$ of the system of collocation equations (17) is non-degenerate for arbitrary *N*.

A similar result is obtained if the restriction operator s_N is chosen in the form

$$(s_N f)_j = \frac{1}{mes\,\delta(y_j)} \int_{\delta(y_j)} f(y)d\Gamma_y \qquad (21)$$

and $r_j(x) = \frac{1}{mes\,\delta(y_j)} \int_{\delta(y_j)} \frac{d\Gamma_y}{|x-y|}$, $j = \overline{1,N}$.

It is obvious that under conditions of Lemma 3 the operator $\mathbf{A}_N^c$, where s_N is defined according to (14)-(15) or (21), or in the case of $f \in C(\Gamma)$ according to (16), is stable.

Consider a discrete analog of condition (20).Let us the quadrature formula

$$\int_\Gamma \varphi(x) r_i(x) d\Gamma_x \approx \sum_{j=1}^N A_j \varphi(x_j) r_i(x_j)\,, \qquad (22)$$

$$x_j \in \Gamma\,, x_j \neq x_i\,, \text{ if } j \neq i\,,$$

is used to calculate the integrals

$$\int_\Gamma \varphi(x) r_i(x) d\Gamma_x\,, \varphi(x) \in U_N\,, i = \overline{1,N}\,,$$

which is exact for integrals

$$\int_\Gamma \varphi(x)\psi(x) d\Gamma_x\,, \varphi(x), \psi(x) \in U_N\,.$$

Consider the system of functions

$$\psi_i(x) = \sum_{k=1}^N \alpha_k^{(i)} \varphi_k(x)\,, \qquad (23)$$

the coefficients $\alpha_k^{(i)}, k,i = \overline{1,N}$, of which we define from *N* systems of linear algebraic equations

$$\sum_{k=1}^N \alpha_k^{(i)} \varphi_k(x_j) = r_i(x_j)\,,\ i,j = \overline{1,N}\,. \qquad (24)$$

Define the conditions under which the functions $\psi_i(x)$, $i = \overline{1,N}$, are linearly independent. From (23) we obtain that

$$\sum_{i=1}^N c_i \psi_i(x) = \sum_{k=1}^N \Big(\sum_{i=1}^N c_i \alpha_k^{(i)}\Big) \varphi_k(x) = 0$$

if and only if

$$\sum_{i=1}^N c_i \alpha_k^{(i)} = 0\,, k = \overline{1,N}\,. \qquad (25)$$

Let us the set of colocation points $X_N = \{y_j\}_{j=1}^N \subset \Gamma$ is chosen in such a way that

$$0 < |x_i - y_i| < \varepsilon\,, d < |x_i - y_j|\,, i,j = \overline{1,N}\,, j \neq i\,, 0 < \varepsilon < \frac{d}{N-1}\,,$$

where $\{x_j\}_{j=1}^N$ are the nodes of quadrature formula (22). Then

$$r_i(x_i) > \sum_{i=1, i\neq j}^N r_i(x_j)\,,$$

matrix $\mathbf{R}_N = \{r_i(x_j)\}_{i,j=1}^N$ due to Hadamard condition isnondegenerateandfrom (24) weobtainthatvectors $\boldsymbol{\alpha}_k = \{\alpha_k^{(j)}\}_{j=1}^N$, $k = \overline{1,N}$, arelinearlyindependent. Hence, equality (25) holds if and only if $c_i = 0$, $i = \overline{1,N}$, i.e. the functions of system $\{\psi_i(x)\}_{i=1}^N$ are linearly independent.

Now, ifthequadratureformula (22) isusedtocalculatetheintegralsincoefficientsofmatrix $\mathbf{A}_N^c$, insteadofthesystemofcollocationequations (14), weactuallysolveasystemwithmatrix

$$\widetilde{\mathbf{A}}_N^c = \{\int_\Gamma \varphi_i(x) \psi_j(x) d\Gamma_x\}_{i,j=1}^N\,,$$

where functions $\psi_i(x)$, $i = \overline{1,N}$, are defined by formulas (23) and (24). The last matrix can be degenerate if and only if there exists a nonzero element $\varphi(x) = \sum_{i=1}^N a_i \varphi_i(x) \in U_N$, orthogonal to all $\psi_i(x)$, $i = \overline{1,N}$, which is impossible, since the system $\{\psi_i(x)\}_{i=1}^N$ forms a basis in the space U_N.

Let us the system of Lagrangian finite elements of the form (6) is used to approximate the unknown potential density $u \in H^m(\Gamma)$ and U_{N_L} is its linear shell. We choose the operators $\tilde{r}_{N_L} : H^m(\Gamma) \to V_{N_L}$ and $\tilde{p}_{N_L} : V_{N_L} \to U_{N_L}$ in the form (7) and (8) respectively and determine the restriction operator $s_{N_L} : H^{m+1}(\Gamma) \to \Phi_{N_L}$ in the form (14), (15). In this case, the system

$$\mathbf{A}_{N_L}^c \mathbf{u}_{N_L} = \mathbf{f}_{N_L}\,,\ \mathbf{A}_{N_L}^c = \tilde{r}_{N_L} A \tilde{p}_{N_L}\,,\ \mathbf{f}_{N_L} = r_{N_L} f\,,$$

implements the collocation method for solution of equation (12). From Lax-Milgram lemma [16] follows that under conditions (20) matrix $\mathbf{A}_{N_L}^c$ is non-degenerate and, accordingly, the definition of operator q_{N_L} in the form $q_{N_L} \mathbf{f}_{N_L} = A \tilde{p}_{N_L} \mathbf{u}_{N_L}$ is correct. Given the left side of inequalities (13), the biectivity of mapping $\tilde{p}_{N_L} : V_{N_L} \to U_{N_L}$, the expressions for norms in the spaces V_{N_L} and Φ_{N_L} in the case $U = H^m(\Gamma)$, $F = H^{m+1}(\Gamma)$ and equality $Q_{N_L} A P_{N_L} u = A P_{N_L} u$, we obtain the validity of inequalities

$$\alpha_m \left\| \mathbf{u}_{N_L} \right\|_{V_{N_L}} \leq \left\| \mathbf{A}^c_{N_L} \mathbf{u}_{N_L} \right\|_{\Phi_{N_L}} \qquad (26)$$

for arbitrary $\mathbf{u}_{N_L} \in V_{N_L}$, in which α_m does not depend on $\mathbf{u}_{N_L}$ .Then from the inequalities (13) and (26), Lemmas2, 3, and basic convergence theorem [17] we obtain the validity of following

Theorem 2. For arbitrary $f \in H^{m+1}(\Gamma)$, m=0,1,…, the approximate solution $u^L_{N_L}$ of equation (12) obtained by collocation method under approximation of unknown potential density by a system of functions constructed on the base of Lagrangian finite elements of m-th degree and the choice of collocation points that satisfies the condition (20) converges to its exact solution, andthereisanestimate

$$\left\| u - u^L_{N_L} \right\|_{H^t(\Gamma)} \leq \frac{\widetilde{C}(1+\beta_t/\alpha_t)}{\alpha_\sigma} h^{\sigma-t} \|f\|_{H^{\sigma+1}(\Gamma)}, \qquad (27)$$

where $0 \leq t \leq \sigma \leq m+1$, $t \leq m$, and h is the maximum area of the grid element on Γ .

IV. Error estimation of approximate solution of the Dirichlet problem for the Laplace equation

Denote

$$v^L_{N_L}(x) = \frac{1}{4\pi} \int_\Gamma \frac{u^L_{N_L}(y)}{|x-y|} d\Gamma_y, \; x \in G, G',$$

and estimate the modulus of value

$$\frac{\partial^\alpha}{\partial x^\alpha}(v(x) - v^L_{N_L}(x)) = \frac{1}{4\pi} \int_\Gamma (u(y) - u^L_{N_L}(y)) \frac{\partial^\alpha}{\partial x^\alpha} \frac{1}{|x-y|} d\Gamma_y,$$

$$x \in G, G', \; \alpha = 0,1,...$$

Let us

$$x \in R^3 \setminus \{\widetilde{x} \in R^3 : |\widetilde{x} - y| < \delta, y \in \Gamma\}. \qquad (28)$$

Using Holder inequality and (24), we obtain

$$\left| \frac{\partial^\alpha}{\partial x^\alpha}(v(x) - v^L_{N_L}(x)) \right| \leq \frac{mes\Gamma}{\delta^{\alpha+1}} \left\| u - u^L_{N_L} \right\|_{L_2(\Gamma)}, \qquad (29)$$

$$x \in G, G', \; \alpha = 0,1,...$$

Then from inequalities (27), (29) and Theorem 2 follow the validity of the next statement.

Theorem3. For arbitrary $f \in H^{m+1}(\Gamma)$, m=0,1,…, an approximate solution of the problem (10), (11) obtained by collocation method under approximation of unknown potential density by systems of functions constructed on the basis of Lagrangian finite elements of the m-th degree, converges to its exact solution, and there is an estimate

$$\left| \frac{\partial^\alpha}{\partial x^\alpha}(v(x) - v^L_{N_L}(x)) \right| \leq \frac{C^*(1+\beta_0/\alpha_0)h^m}{\alpha_m \delta^{\alpha+1}} \|f\|_{H^{m+1}(\Gamma)},$$

$$x \in G, G', \; \alpha = 0,1,...$$

Conclusions

The paper describes the conditions and evaluations of convergence of collocation method for solution of Fredholm integral equation of the first kind for the simple layer potential in case of closed boundary surface in a three-dimensional space. Approximation of potential density was performed using Lagrangian finite elements of various orders on rectangular grids constructed in the desired function definition domain. Estimations were obtained for the error of approximate solution of Dirichlet problem for Laplace equation that is equivalent to the integral equation for the simple layer potential. The approach proposed can be used to define convergence of collocation method for solving potential theory integral equations that are equivalent to the boundary value problems for equations of mathematical physics and other types of finite elements of various orders, constructed on both rectangular and triangular grids in desired potential density definition domain.

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